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DIFFERENTIABLE RIGIDITY FOR HYPERBOLIC TORAL ACTIONS

BY

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ABSTRACT

Our aim is to extend existing results about differentiable rigidity of higher rank abelian actions by automorphisms of a torus. Previous proofs have required an assumption of *semisimplicity*, that is, that the action is by commuting diagonalizable matrices. Here we introduce a technique that utilizes the unipotent part of a non-semisimple action, which allows us to discard the semisimplicity assumption. In its place we will make a technical assumption that the spectrum of the action restricted to leaves of the coarse Lyapunov decomposition is sufficiently narrow.

1. Introduction

In 1986, Zimmer conjectured that there should be local differentiable rigidity for a large class of actions of higher rank Lie groups and lattices in these. This initiated a program of study, with a series of results by Hurder, Katok, Lewis, Zimmer, Spatzier, Goetze, Qian, and others. An early observation was that many important actions, like that of $\mathrm{SL}(k,\mathbb{Z})$ on $\mathbb{T}^k$, contain an action of $\mathbb{Z}^k$, $k \geq 2$, and that such abelian actions are themselves enough to imply rigidity. This paper continues in that tradition, studying Anosov actions of $\mathbb{Z}^k$ on the torus $\mathbb{T}^d$.

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A variety of techniques have been brought to bear on the problem of rigidity of Anosov actions (or other actions with hyperbolic properties). Our paper will focus on geometric techniques, studying geometric structures associated to an Anosov action. This builds on the works of Katok and Lewis [KL91] and Katok and Spatzier [KS97]. This is in contrast to the super-rigidity approaches of, e.g., Katok, Lewis, and Zimmer [KLZ96], Qian and Yue [QY98], or the recent KAM method results of Damjanovic and Katok [DK04].

We recall the standard terminology that an action $\alpha\colon \mathbb{Z}^k \to \mathrm{SL}(d,\mathbb{Z})$ of $\mathbb{Z}^k$ on the d-dimensional torus $\mathbb{T}^d$ by toral automorphism is Anosov if there is some $\mathbf{n} \in \mathbb{Z}^k$ such that $\alpha(\mathbf{n})$ is an Anosov diffeomorphism, i.e., that no eigenvalue of $\alpha(\mathbf{n})$ has modulus one. Throughout this paper, we will hold fixed a generating set of $\mathbb{Z}^k$ of vectors $\mathbf{e}_1,\ldots,\mathbf{e}_k$. An action $\beta\colon \mathbb{Z}^k \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ is C^1 close to α if each $\beta(\mathbf{e}_i)$ is close to $\alpha(\mathbf{e}_i)$ in the C^1 topology. We say that β is conjugate to α if there exists a homeomorphism $\phi\colon \mathbb{T}^d \to \mathbb{T}^d$ such that $\phi\circ\alpha(\mathbf{n})(p) = \beta(\mathbf{n})\circ\phi(p)$ for every $p \in \mathbb{T}^d$ and $\mathbf{n} \in \mathbb{Z}^k$. We say that action α is locally differentiably rigid if every C^∞ action β which is sufficiently C^1 close to α is conjugate to α by a C^∞ map ϕ.

We say that α has a virtually cyclic factor γ if there exists an α-invariant proper subtorus $Y \subset \mathbb{T}^d$ such that γ is the induced action on $\mathbb{T}^{d'} \cong \mathbb{T}^d/Y$, $d' \geq 1$, for which $\gamma\circ\pi = \pi\circ\alpha$ and $\gamma(\mathbb{Z}^k)$ contains a cyclic subgroup of finite index.

Our main result is the following:

THEOREM 1.1: *Suppose α is an Anosov linear action of $\mathbb{Z}^k$ on $\mathbb{T}^d$ which has no virtually cyclic factors. Assume that α has narrow spectrum within each coarse Lyapunov leaf (see Section 3.5). Then α is locally differentiably rigid.*

The assumption that there are no virtually cyclic factors is necessary to ensure that we are in a genuinely higher rank situation. There is a profound difference between rank 1 and higher rank actions. An action by $\mathbb{Z}$ cannot possibly be rigid; the differential at a fixed point, for instance, is preserved under smooth conjugacy but not under perturbation. This is just as true if there is any factor of the action on which the action is essentially rank 1.

The assumption that α has narrow spectrum within a coarse Lyapunov leaf is technical. It allows us to work with linear structures, like vector bundles, rather than more complicated polynomial objects. We discuss this further in Section 3.5.

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2. Outline of methods

Our method proceeds by building up geometric information for both the linear and perturbed actions. We have from structural stability a candidate conjugacy ϕ which is a priori only (Hölder) continuous. We gain more information about the smoothness of ϕ by producing matching geometric structures on the two conjugated sides. For instance, the stable and unstable manifolds for an element of the linear action will be conjugated to the stable and unstable manifolds for the corresponding element of the perturbed action. This holds for intersections of stable and unstable manifolds as well: the maximal such intersections form what is known as the coarse Lyapunov decomposition (see Section 3.3).

We need to decompose further than this to identify the geometric structures which come from the shearing from non-semisimple linear parts of the action. We introduce the notion of logarithmic directions which allow us to identify what we call the finer dynamical filtration. This is a refinement of the coarse Lyapunov decomposition on the leaves of which we can isolate isometric behavior of the action.

We build up smoothness of ϕ on this filtration using a number of tools. The first of these is the nonstationary normal forms theory which allows us to linearize the perturbed action in a certain family of coordinates. The natural statement of this theory is within a certain bundle structure, which we describe in Section 3.4. The next step is to study the closure of the orbit of a point chosen near a fixed point of the action. This orbit closure step shows that a certain functional equation for the $\mathbb{Z}^k$ actions actually implies a similar functional equation for $\mathbb{R}$ actions. This gives smoothness of the conjugacy along a family of curves. Using elliptic operator theory, the smoothness of ϕ along leaves of a sufficient family of foliations implies smoothness of the restriction of ϕ on the foliated manifold. This argument is presented in Section 5. Finally, we show how these steps can be applied inductively to show that smoothness of ϕ along the coarse Lyapunov foliations can be built up inductively from one level of the dynamical filtration to the next. This induction is carried out in Section 6, where we also use elliptic operator theory to conclude smoothness of ϕ.

3. Preliminaries

3.1. Structural Stability. A diffeomorphism T (or $\mathbb{Z}^k$ action α) is said to be structurally stable if whenever another diffeomorphism S is sufficiently C^1 close to T (or $\mathbb{Z}^k$ action β is sufficiently C^1 close to α) then there exists a homeomorphism ϕ such that $S = \phi \circ T \circ \phi^{-1}$ (or $\beta = \phi \circ \alpha \circ \phi^{-1}$). The structural stability of an Anosov diffeomorphism is a classical result, due to Anosov. It is furthermore true (see, e.g., [KH95]) that there is a unique such ϕ close to the identity in the C^0 topology.

As is known this quickly implies structural stability for the α action. To see this, fix some $\mathbf{n}$ such that $T = \alpha^{\mathbf{n}}$ is Anosov, and let β be a perturbation of α which is sufficiently small in the C^1 topology. Let ϕ be the conjugacy such that $\alpha^{\mathbf{n}} \circ \phi = \phi \circ \beta^{\mathbf{n}}$. Then for a generator $\beta^{\mathbf{e}_j}$, the conjugate image $\phi \circ \beta^{\mathbf{e}_j} \circ \phi^{-1}$ commutes with $\alpha^{\mathbf{n}}$ and is close to $\alpha^{\mathbf{e}_j}$ in the C^0 topology. Therefore, $\alpha^{-\mathbf{e}_j} \circ \phi \circ \beta^{\mathbf{e}_j} \circ \phi^{-1}$ also commutes with $\alpha^{\mathbf{n}}$ and is close to the identity in the C^0 topology; by the above-mentioned uniqueness of conjugacies this homeomorphism must be the identity. This implies that $\phi \circ \beta^{\mathbf{e}_j} \circ \phi^{-1} = \alpha^{\mathbf{e}_j}$ for all j. Thus the same conjugacy works for the entire action, so the action is structurally stable.

3.2. Reduction to the Rank Two Case. Virtually cyclic factors for α have also been called rank one factors [KS97] and the assumption that there are none has several equivalent formulations.

Proposition 3.1: *Let α be a $\mathbb{Z}^k$-action by automorphisms of $\mathbb{T}^d$. Then the following are equivalent.*

(1) *α has no virtually cyclic factors.*
(2) *For every common eigenvector v of the linear action associated to α there exists a subgroup of $\mathbb{Z}^k$ isomorphic to $\mathbb{Z}^2$ such that the eigenvalue to v is not equal to one for any nontrivial element of the subgroup.*
(3) *There exists a subgroup of $\mathbb{Z}^k$ isomorphic to $\mathbb{Z}^2$ such that any non-trivial element of the subgroup acts ergodically on $\mathbb{T}^d$ with respect to the Haar measure.*

Although this is known, see [Sta99], we give the short proof for the sake of completeness.

Proof: Suppose (2) does not hold. Then there exists a common eigenvector $v \in \mathbb{R}^d$ and linearly independent elements $\mathbf{n}_1, \ldots, \mathbf{n}_{k-1} \in \mathbb{Z}^k$ such that $\alpha^{\mathbf{n}_i} v = v$ for all i. Since the linear maps $\alpha^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{Z}^k$ commute, we can split $\mathbb{R}^d$ into finitely many generalized eigenspaces and furthermore simultaneously triangulate these matrices within the eigenspaces (where we allow for two-by-two blocks on the diagonal to accommodate complex eigenvalues). Since v is in the common kernel of $\alpha^{\mathbf{n}_i} - \mathrm{Id}$ for $i = 1, \dots, k-1$, within at least one of the generalized eigenspaces the diagonal entries of these matrices are zero. It then follows that the images of the maps $\alpha^{\mathbf{n}_i} - \mathrm{Id}$ for $i = 1, \dots, k-1$ span a proper rational subspace $V \subset \mathbb{R}^d$. Clearly V defines a proper α-invariant subtorus Y of $\mathbb{T}^d$, and α induces an action γ on $\mathbb{T}^d/Y$. However, then $\gamma^{\mathbf{n}_i}$ acts trivially and so γ is a virtually cyclic factor of α, i.e., (1) does not hold.

Assume now (2) holds. For every eigenvector v we define

$$\Lambda_v = \{\mathbf{n} \in \mathbb{Z}^k : \text{the eigenvalue of } \alpha^{\mathbf{n}} \text{ for } v \text{ equals one}\}.$$

Then by assumption this subgroup has dimension less than or equal to $k-2$ for every v. It is easy to pick a subgroup of $\mathbb{Z}^k$ isomorphic to $\mathbb{Z}^2$ that intersects any Λ_v trivially. Since an automorphism of a torus acts ergodically if and only if no eigenvalue is a root of unity, it follows that this subgroup satisfies (3).

It is clear that (3) implies the same for any factor of $\mathbb{T}^d$, so that there can be no virtually cyclic factors. ∎

Corollary 3.2: *Let α be a $\mathbb{Z}^k$-action by automorphisms of $\mathbb{T}^d$ without virtually cyclic factors. Then there exists a $\mathbb{Z}^2$-subaction that also has no virtually cyclic factors. For a non-identity element of that subaction, no eigenvalue to a common eigenvector is a root of unity.*

In the remainder of the paper we will work with a $\mathbb{Z}^2$-subaction as in the above corollary.

3.3. Coarse Lyapunov Decomposition. This is just as in [KS97]. Let α denote the linear $\mathbb{Z}^2$ action on $\mathbb{R}^d$ as well as the original action on $\mathbb{T}^d$. Let β be a perturbation sufficiently C^1 close such that ϕ as in Section 3.1 exists (one more restriction of the closeness of the perturbation will be given when we discuss the narrow spectrum assumption in Section 3.5). For any common (generalized) eigendirection $x \in \mathbb{R}^d$ of α we define $\chi_x(\mathbf{n})$ to be the logarithm of the absolute value of the eigenvalue of $\alpha^{\mathbf{n}}$ for x. Clearly $\chi_x\colon \mathbb{Z}^2 \to \mathbb{R}$ extends to a linear functional on $\mathbb{R}^2$ — the Lyapunov weight. Let $V_\chi \subseteq \mathbb{R}^d$ be the Lyapunov subspace that is generated by all generalized eigenspaces that give rise to the same Lyapunov weight χ. For the corresponding half-space $\mathcal{H} = \{v \in \mathbb{R}^2 | \chi(v) < 0\}$ bounded by a Lyapunov hyperplane $\partial\mathcal{H}$, we associate the foliation $V^-_{\mathcal{H}}$ consisting of the intersections of stable manifolds for all elements of the action in that half-space. Alternatively, $V^-_{\mathcal{H}}$ is the foliation $\{x + \bigoplus_{\chi'} V_{\chi'} | x \in \mathbb{T}^d\}$ where the sum is over weights χ' which are positively proportional to χ. This is a foliation with smooth leaves, which is conjugate under ϕ to a foliation $W^-_{\mathcal{H}}$ consisting of the intersections of stable manifolds for the corresponding elements of the β action. The stable manifolds are smooth immersed submanifolds; consequently, these foliations will have leaves which are immersed copies of $\mathbb{R}^k$ for some dimension k.

Example 3.3: Let

$$C_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix};$$

then both define automorphisms of $\mathbb{T}^4$. One can check that $C_1C_2 = C_2C_1$, so that they define a $\mathbb{Z}^2$-action γ on $\mathbb{T}^4$ by automorphisms. It is easy to check that the eigenvalues for common eigenvectors to C_1 and C_2 are precisely

$$(1+\sqrt{2}, 2+\sqrt{3}), (1+\sqrt{2}, 2-\sqrt{3}), (1-\sqrt{2}, 2-\sqrt{3}), (1-\sqrt{2}, 2+\sqrt{3}).$$

Note that in this list every pair consists of multiplicatively independent numbers. By Proposition 3.1, γ has no virtually cyclic factors.

With this we can give an example of a non-semi-simple $\mathbb{Z}^2$-action that has no virtually cyclic factors. Define

$$A_1 = \begin{pmatrix} C_1 & \mathrm{Id} \\ & C_1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} C_2 & \\ & C_2 \end{pmatrix}.$$

It is clear that the eigenvalues of A_1, A_2 are the same as for C_1, C_2, and it is easy to check that $A_1A_2 = A_2A_1$. Therefore, A_1, A_2 define a $\mathbb{Z}^2$-action α by hyperbolic automorphisms of $\mathbb{T}^8$. Again by Proposition 3.1, α has no virtually cyclic factors.

Let $E \subset \mathbb{R}^4$ be a common eigenspace of C_1 and C_2, say for the eigenvalues $(1+\sqrt{2}, 2+\sqrt{3})$. Then E is one-dimensional; let $\mathbf{e} \in E$ be a generator. Then $\kappa = E \times E \subset \mathbb{R}^8$ is a generalized common eigenspace for A_1 and A_2 and the eigenvalues $(1+\sqrt{2}, 2+\sqrt{3})$. Define the basis $v_1 = (\mathbf{e}, 0)$ and $v_2 = (0, \mathbf{e})$; then in this basis the two linear maps are

$$A_1|_\kappa = \begin{pmatrix} 1+\sqrt{2} & 1 \\ & 1+\sqrt{2} \end{pmatrix} \quad \text{and} \quad A_2|_\kappa = \begin{pmatrix} 2+\sqrt{3} & \\ & 2+\sqrt{3} \end{pmatrix}.$$

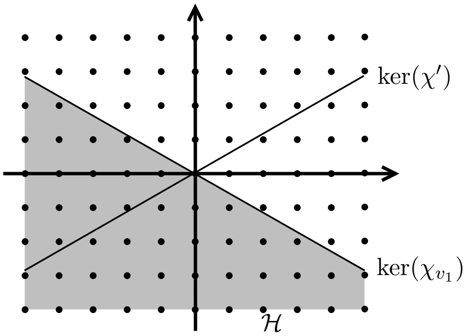

Figure 1. The halfspace $\mathcal{H}$ is bounded by the kernel of a Lyapunov weight. The kernel of the non-proportional weights is also shown.

As v_1 is a common eigenvector of A_1 and A_2, there is an associated Lyapunov weight χ_{v_1} that maps $\mathbf{n} = (n_1, n_2)$ to the log of the eigenvalue of $A_1^{n_1} A_2^{n_2}$ for v_1, namely $n_1 \log(1+\sqrt{2}) + n_2 \log(2+\sqrt{3})$. The other Lyapunov weights map $\mathbf{n}$ to $\pm n_1 \log(1+\sqrt{2}) + \pm n_2 \log(2+\sqrt{3})$ with the four choice of plus or minus giving the four different weights. The kernel of χ_{v_1} divides the plane into two halfspaces, one on which v_1 is expanded by each element of the α action and one on which it is contracted by each element. This kernel coincides with the kernel of the Lyapunov weight that is negatively proportional to χ_{v_1}. Let $\mathcal{H}$ be the halfspace on which χ_{v_1} is negative (Figure 1). The coarse Lyapunov foliation $V_{\mathcal{H}}^-$ is the intersection of the stable foliations of each $\alpha^{\mathbf{n}}$ with $\mathbf{n} \in \mathcal{H}$. This corresponds to the intersection of generalized eigenspaces for $A_1^n A_2^m$ for eigenvalues whose magnitude is less than 1, or equivalently, for Lyapunov weights with $\chi'(\mathbf{n}) < 0$. The Lyapunov weights not proportional to χ_{v_1} are positive for some $\mathbf{n} \in \mathcal{H}$ and negative for others, while the weight negatively proportional to χ_{v_1} is always positive for $\mathbf{n}$ in $\mathcal{H}$. Thus the subspace of $\mathbb{R}^8$ corresponding to the coarse Lyapunov foliation $V_{\mathcal{H}}^-$ of $\mathbb{T}^8$ is κ.

3.4. E-bundles. The natural setting for the nonstationary subresonance normal form theory developed by Guysinsky and Katok ([GK98, Guy02]) is that of maps on a bundle over the base manifold. It will be useful for us to work within this same structure. One family of bundles we are particularly interested in comes directly from foliations of the manifold. Given a foliation V of a manifold M with smooth leaves $V(x)$ which are immersed copies of $\mathbb{R}^k$, consider the tangent bundle to V as a bundle $\tilde{V}$ over M with fibers that are copies of $\mathbb{R}^k$. For a fixed Riemannian metric on M we induce a Riemannian metric on each leaf $V(p)$. Let $\exp_p\colon \tilde V(p) \to V(p)$ be the exponential map with respect to this induced metric. Then a map $f\colon M \to M$ which maps leaves of V to leaves of V lifts naturally to a map defined on a neighborhood of the zero section of $\tilde V$ by $\tilde f(p,x) = (f(p), \exp^{-1}_{f(p)} \circ f \circ \exp_p(x))$. Such a bundle $\tilde V$ has additional structure. The exponential map from a fiber $\tilde V(x)$ to leaf $V(x)$, composed with the inverse of the exponential map at a nearby point $y \in V(x)$, gives a smooth map from a neighborhood of zero in $\tilde V(x)$ to $\tilde V(y)$. This is the motivating example of what we will call an e-bundle, though we will prefer to work with the full fiber rather than a neighborhood of zero. We will discuss this example further in 3.5 and 4.2.

Definition 3.4 (e-bundle)*:* An *e-bundle over the foliation V of a manifold M* is a vector bundle B over M together with smooth maps $\Theta_{qp}\colon B(p) \to B(q)$ for every pair of points p and q in the same leaf of V (Figure 2) satisfying (i) $\Theta_{uq} \circ \Theta_{qp} = \Theta_{up}$ whenever p, q, and u are in the same leaf and (ii) $\Theta_{pp} = Id$ for every p.

Note that we do not require the bundle fibers to be identified with the leaves of the foliation; in our main argument this will not be the case. We will need to take an e-bundle which arises from a foliation and consider both sub-bundles and quotients of this e-bundle. See Section 4.2–4.3.

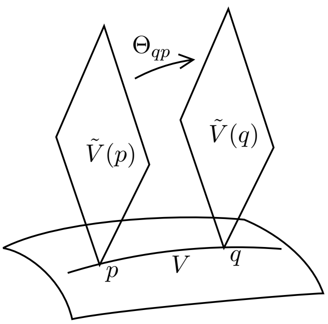

Figure 2. The map Θ_{qp} maps from one fiber of the e-bundle to another fiber.

Definition 3.5 (e-bundle homomorphism)*:* Suppose B_i is an e-bundle over a foliation V_i of M_i, $i = 1, 2$. A continuous map $\phi\colon B_1 \to B_2$ is an *e-bundle homomorphism* if it maps fibers of B_1 into fibers of B_2, maps the zero section of B_1 into the zero section of B_2, maps leaves of V_1 into leaves of V_2 (where M_i is identified with the zero section of B_i), and satisfies for all p and q in the same leaf of V_1 that

$$\Theta^{B_2}_{\phi(q)\phi(p)} \circ \phi|_{B_1(p)} = \phi|_{B_1(q)} \circ \Theta^{B_1}_{qp}. \tag{3.1}$$

We say that ϕ is *smooth* if the restriction to every fiber $\phi|_{B_1(p)}$ is smooth; similarly we say ϕ is *linear* if every restriction to a fiber is a linear map.

3.5. Nonstationary Normal Forms. Let α be a $\mathbb{Z}^2$-action by automorphisms of a torus $\mathbb{T}^d$, and let β be a smooth perturbation sufficiently close to α. Let $V^-_{\mathcal{H}}$, and $W^-_{\mathcal{H}}$ be as in Section 3.3. For an element $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathcal{H}$, the space $W^-_{\mathcal{H}}$ is contracted. For a sufficiently small (in the C^1 topology) perturbation of the α action, the perturbation β satisfies the conditions of Theorem 1.2 of [GK98]. This guarantees that there is a family of local coordinate systems in the leaves of $W^-_{\mathcal{H}}(x)$, depending continuously on x, in which $\beta^{\mathbf{n}}$ is given by a subresonance polynomial map.

Definition 3.6 (narrow spectrum)*:* We say that F has *narrow spectrum* if there exists λ larger than the spectral radius of F and μ larger than the spectral radius of F^{-1} with $\lambda^2\mu < 1$ (see Figure 3).

For the proof of the main theorem, we assume that the entire spectrum of both $\alpha^{\mathbf{n}}$ on $W^-_{\mathcal{H}}$ and the perturbation $\beta^{\mathbf{n}}$ on $W^-_{\mathcal{H}}$ satisfies this narrow spectrum assumption. This property for the spectrum of $\beta^{\mathbf{n}}$ is inherited from the corresponding property for $\alpha^{\mathbf{n}}$ provided that $\beta^{\mathbf{n}}$ is a small enough perturbation in the C^1 topology (see [Pes04, Theorem 3.4]). The narrow spectrum assumption tells us, in the terminology of [GK98], that there are no nontrivial subresonance relations, which implies that in the normal form coordinates the map $\beta^{\mathbf{n}}$ is linear. Furthermore, the entire β action is by linear maps in these coordinates ([GK98] Theorem 1.3) and the coordinates vary smoothly in the leaves of $W^-_{\mathcal{H}}$ ([GK98] Theorem 2.1) while only continuously in the whole of $\mathbb{T}^d$. (Below we will use the e-bundle terminology to give a notation for these normal form coordinates.)

The condition of no integer resonances for the (restricted to a Coarse Lyapunov leaf) linear part of the α action implies that there are no actual resonances for the spectrum of a sufficiently small perturbation of the action. This presumably implies that the nonstationary normal form could be linearized, though this version of the theory has not been published. Thus we believe it would be sufficient to assume for the α action that among a set of positively proportional Lyapunov weights, no weight is a positive integer combination of the other weights.

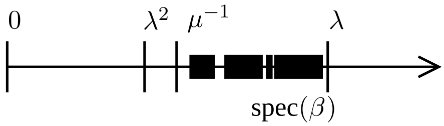

Figure 3. The Mather spectrum of $\beta^{\mathbf{n}}$ restricted to $W_{\mathcal{H}}^-$ is assumed to be narrow.

Clearly, for the linear action α on $\mathbb{T}^d$ the bundle $\tilde{V}_{\mathcal{H}}^-$ over the foliation $V_{\mathcal{H}}^-$ is an e-bundle. Furthermore, since α preserves the foliation $V_{\mathcal{H}}^-$ we can lift α trivially to $\tilde{\alpha}$ which defines an action by linear e-bundle homomorphisms on $\tilde{V}_{\mathcal{H}}^-$.

We now construct an e-bundle $\tilde{W}_{\mathcal{H}}^-$ and extend β to an action $\tilde{\beta}$ of linear e-bundle homomorphisms of $\tilde{W}_{\mathcal{H}}^-$. The bundle $\tilde{W}_{\mathcal{H}}^-$ is the tangent bundle to the foliation $W_{\mathcal{H}}^-$; we need to define the maps Θ_{pq} for any two points p, q in the same $W_{\mathcal{H}}^-$-leaf. Let e_p be the composition of the exponential map $\exp_p$ from fibers of $\tilde{W}_{\mathcal{H}}^-$ with the coordinate change map given by the nonstationary normal forms theory: e_p is defined and smooth on a neighborhood of $0 \in \tilde{W}_{\mathcal{H}}^-(p)$. Moreover, the resulting map $e(p,x) = e_p(x)$ defined on a neighborhood of the zero section of the bundle is continuous (by the continuity of the normal form coordinates). As we discussed above, we can define the lift of β on the neighborhood of the zero section by conjugation with the exponential map; in the new coordinates this is

$$\tilde{\beta}^{\mathbf{n}}(p,x) = (\beta^{\mathbf{n}}p,\ e_{\beta^{\mathbf{n}}p}^{-1}(\beta^{\mathbf{n}}(e_p(x)))). \tag{3.2}$$

Then this $\tilde{\beta}$ action is linear in the fibers where defined (this is the point of the non-stationary normal forms theory). We extend it by linearity to be defined on the whole bundle. Note that the resulting action, which we still denote $\tilde{\beta}$, is still an action. Also, since β is continuous and $e(p,\cdot)$ is injective for any fixed p it follows easily that $\tilde{\beta}$ is also continuous, i.e., the linear map in the local coordinates depends continuously on the base point.

Proposition 3.7: *There is an extension of e_p to an immersion of $\tilde{W}_{\mathcal{H}}^-(p)$ onto $W_{\mathcal{H}}^-(p)$. Together these define a continuous map e: $\tilde{W}_{\mathcal{H}}^- \to \mathbb{T}^d$ that satisfies $\beta^{\mathbf{n}} \circ e = e \circ \tilde{\beta}^{\mathbf{n}}$.*

Proof: We use (3.2) to define e. That is, given a point $p \in \mathbb{T}^d$ and an $x \in \tilde{V}(p)$, choose an $\mathbf{n} \in \mathcal{H}$ such that $\tilde{\beta}^{\mathbf{n}}(p,x)$ is in the domain of definition of

$e_{\beta^{\mathbf{n}}(p)} = e(\beta^{\mathbf{n}}, \cdot)$. We define $e(p,x) = \beta^{-\mathbf{n}} e(\tilde{\beta}^{\mathbf{n}}(p,x))$. This coincides with the old definition of e_p when both are defined, as in this case $\tilde{\beta}^{\mathbf{n}}(p,\cdot) = e^{-1}_{\beta^{\mathbf{n}}p} \circ \beta^{\mathbf{n}} \circ e_p$. This also shows that the definition does not depend on $\mathbf{n}$. Continuity of e follows from continuity of the coordinates in the normal form theory. ∎

Corollary 3.8: *The bundle $\tilde{W}^-_{\mathcal{H}}$ can be given the structure of an e-bundle, where $\Theta_{qp} = e_q^{-1} \circ e_p$ for $p, q \in \mathbb{T}^d$ within the same $W^-_{\mathcal{H}}$-leaf. Furthermore, the map ϕ lifts to an e-bundle isomorphism $\tilde{\phi}(p,x) = (\phi(p), e^{-1}_{\phi(p)}(\phi(p+x)))$ between $\tilde{V}^-_{\mathcal{H}}$ and $\tilde{W}^-_{\mathcal{H}}$ which conjugates $\tilde{\alpha}$ to $\tilde{\beta}$.*

Proof: As indicated in Section 3.4 the map Θ_{qp} defined in the corollary gives $\tilde{W}^-_{\mathcal{H}}$ the structure of an e-bundle. The continuity of $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ in a neighborhood of the zero section follows from the continuity of $e_{\phi(p)}$ and the fact that the coordinates of the nonstationary normal forms depend continuously on the base point. However, then by (3.2) $\tilde{\phi}$ must be a homeomorphism. Finally, (3.1) follows from the definitions

$$\begin{aligned}
\Theta_{\phi(q)\phi(p)} \circ \tilde{\phi}(p,x) &= \Theta_{\phi(q)\phi(p)}(\phi(p), e^{-1}_{\phi(p)}(\phi(p+x))) \\
&= (\phi(q), e^{-1}_{\phi(q)}(\phi(p+x)) \\
&= (\phi(q), e^{-1}_{\phi(q)}(\phi(q+w+x)) \\
&= \tilde{\phi}(q, w+x) = \tilde{\phi} \circ \Theta_{qp}(p,x).
\end{aligned}$$

Here w is such that $p = q + w$ and is uniquely determined by the requirement that it belongs to the linear subspace that defines $V^-_{\mathcal{H}}$. ∎

4. Logarithmic sequences

4.1. Logarithmic sequences for α. We will be interested in studying closely the behavior of the actions near a fixed point for a finite index subaction; this study will also be restricted to leaves of the coarse Lyapunov foliation. This leads to a study of the following construction for the linear action α.

Consider a leaf of $V^-_{\mathcal{H}}$ through a fixed point of the α action. This is an immersed copy of $\mathbb{R}^{d_{\mathcal{H}}}$, where $d_{\mathcal{H}} = \dim(V^-_{\mathcal{H}})$. Denote the restricted action of α on this $\mathbb{R}^{d_{\mathcal{H}}}$ as $\alpha_{\mathcal{H}}$. This is a linear action generated by commuting matrices.

Suppose $\alpha^{\mathbf{k}}_{\mathcal{H}}$ is a contracting matrix. The behavior of the action as we move parallel to the boundary $\partial\mathcal{H}$ of the half-space will in general involve shearing which takes place at a polynomial rate. By composing these shearing maps with an appropriate power of the contracting matrix $\alpha^{\mathbf{k}}_{\mathcal{H}}$ we can reduce the maximal entry of the matrix to size approximately one. More precisely, the maximal entry can be forced to have size less than one, but such that when $\alpha_{\mathcal{H}}^{-\mathbf{k}}$ is applied the maximal entry has size larger than one; thus such matrices belong to a compact family that does not contain the zero matrix. As the contraction takes place at exponential speed while the shearing is at a polynomial speed, the appropriate power of the contraction is like the logarithm of the distance along $\partial\mathcal{H}$. We call such a sequence of elements of the $\alpha_{\mathcal{H}}$ action a *logarithmic sequence* if the matrices converge to a nonzero matrix with nontrivial kernel. The discussion above shows the existence of a convergent sequence of elements with nonzero limit. The kernel will be nontrivial unless there is no shearing, in which case the sequence remains within a bounded distance of $\partial\mathcal{H}$.

Suppose that $\alpha_{\mathcal{H}}^{\mathbf{n}_i}$ is a logarithmic sequence for the $\alpha_{\mathcal{H}}$ action. Let A be the limit of the logarithmic sequence. Note that A commutes with $\alpha_{\mathcal{H}}$, so the kernel of A is a subspace which is invariant under the $\alpha_{\mathcal{H}}$ action. Then we can restrict the $\alpha_{\mathcal{H}}$ action to the kernel of A and obtain another action which either has no shearing or itself possesses a logarithmic sequence. In this way we produce a filtration which we will call the *logarithmic filtration* of the linear space $\mathbb{R}^{d_{\mathcal{H}}}$,

$$\kappa_0 \subset \kappa_1 \subset \cdots \subset \kappa_\ell = \mathbb{R}^{d_{\mathcal{H}}},$$

where on κ_0 there is no shearing along the $\partial\mathcal{H}$ direction and each κ_j is the kernel of the limit A_{j+1} of the logarithmic sequence for the action restricted to κ_{j+1}.

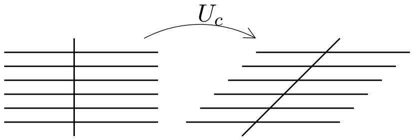

Figure 4. The map U_c will in general have polynomial shearing.

It will be useful to us to understand the shearing along $\partial\mathcal{H}$ using a unipotent matrix $U^{\mathbf{c}}$. Let $U_1 = \mathrm{diag}(\alpha_{\mathcal{H}}^{\mathbf{e}_1})^{-1}\alpha_{\mathcal{H}}^{\mathbf{e}_1}$ and $U_2 = \mathrm{diag}(\alpha_{\mathcal{H}}^{\mathbf{e}_2})^{-1}\alpha_{\mathcal{H}}^{\mathbf{e}_2}$, where $\mathrm{diag}(\cdot)$ denotes the semisimple part of the matrix in question. These are commuting unipotent matrices and can be simultaneously put into upper-triangular form. Take a nonzero $\mathbf{c} = (b_1, b_2) \in \partial\mathcal{H}$ (which is not necessarily in $\mathbb{Z}^2$) and define the unipotent matrix for the critical direction (see Figure 4)

$$U^{\mathbf{c}} = U_1^{c_1} U_2^{c_2} = \exp(c_1 \log U_1 + c_2 \log U_2),$$

where the matrix logs are defined since U_i are unipotent. Similarly, we can define $U^{\mathbf{m}}$ for any $\mathbf{m} \in \mathbb{R}^2$.

LEMMA 4.1: *The eigenspace of $U^{\mathbf{c}}$ is equal to κ_0. Furthermore, the image of A_j is contained in κ_0 for any $j = 1, 2, \ldots, \ell$.*

The above construction and the proof of the lemma below will be made more clear by considering first some examples.

Example 4.2: For the matrices $A_1|_\kappa$ and $A_2|_\kappa$ in Example 3.3, we get

$$U_1 = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{and} \quad U_2 = \mathrm{Id}\,.$$

Taking $\mathbf{c} = (1, -\log(1+\sqrt{2})/\log(2+\sqrt{3}))$, we obtain $U^{\mathbf{c}} = U_1$. Then

$$U^{s\mathbf{c}} A_2^t|_\kappa = (2+\sqrt{3})^t \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix},$$

so to control the size of the largest entry of this matrix, we want to have $s(2+\sqrt{3})^t \approx 1$. This curve of real s and t where this holds exactly is $t = -\log(s)/\log(2+\sqrt{3})$. For (s,t) within a bounded distance of this curve, $s(2+\sqrt{3})^t$ is within a bounded distance of 1. A logarithmic sequence consists of lattice points $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ which approximate the curve $s\mathbf{b} + t\mathbf{e}_2$ to within bounded distance, and for which $A_1^{n_1}|_\kappa A_2^{n_2}|_\kappa$ converge. The limiting matrix will be $A = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, where a is positive. The kernel of A is κ_0 and the two-dimensional κ is κ_1.

Example 4.3: Suppose now that

$$U^{\mathbf{c}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

Then $\kappa_0 = \mathbb{R}^2 \times \{0\}^2 \subset \mathbb{R}^4 = V_{\mathcal{H}}^-$. Note that from $U^{\mathbf{c}}$ alone we cannot determine κ_1. With this $U^{\mathbf{c}}$, for $\alpha_{\mathcal{H}}^{\mathbf{k}} = (2-\sqrt{3})\,\mathrm{Id}$ we get $\kappa_1 = \mathbb{R}^4$ as in the example above, while if

$$\alpha_{\mathcal{H}}^{\mathbf{k}} = \begin{pmatrix} 2-\sqrt{3} & 1 & & \\ & 2-\sqrt{3} & & \\ & & 2-\sqrt{3} & 1 \\ & & & 2-\sqrt{3} \end{pmatrix}$$

then a similar consideration shows $\kappa_1 = \mathbb{R}^3 \times \{0\}$ and $\kappa_2 = \mathbb{R}^4$.

Proof of Lemma 4.1: Let $\mathbf{m} = (m_1, m_2)$ be an element of $\mathcal{H}$, so $\mathbf{c}$ and $\mathbf{m}$ give another basis for $\mathbb{R}^2$. If $\mathbf{n} = s\mathbf{c} + t\mathbf{m} \in \mathbb{Z}^2$ is within bounded distance of $\partial\mathcal{H}$

then $\alpha^{\mathbf{n}}_{\mathcal{H}} = \operatorname{diag} \alpha^{\mathbf{n}}_{\mathcal{H}} U^{t\mathbf{m}} U^{s\mathbf{c}}$ is, up to the power of $U^{\mathbf{c}}$, from a compact family of maps. This shows that an element v is in the eigenspace of $U^{\mathbf{c}}$ if and only if $\alpha^{\mathbf{n}}_{\mathcal{H}} v$ stays bounded as $\mathbf{n} \to \infty$ along $\partial\mathcal{H}$. For if v is in the eigenspace then $U^{\mathbf{c}} v = v$, and if v is not in the eigenspace then $U^{s\mathbf{c}} v$ grows polynomially with s. This shows the first claim.

Now suppose $\alpha^{\mathbf{n}_i}_{\mathcal{H}} \to A_j$ is a logarithmic sequence on one of the subspaces κ_j defined above. We restrict our linear maps to κ_j and without loss of generality assume that we have chosen a basis such that the matrices are of block form where each block corresponds to a particular (complex pair of) eigenvalue(s) and the matrices are upper triangular. In each such block we also use an extension of a basis of κ_0 to the whole subspace, so that the first rows in every block correspond to κ_0. Notice that an entry of the matrix $U^{s\mathbf{c}}$ is some polynomial function $p(s)$ of s where the maximal degree that appears equals the dimension of the maximal Jordan block minus one and that in each block the maximal degree only appears in the rows corresponding to κ_0. Then an entry of $\alpha^{\mathbf{n}_i}_{\mathcal{H}}$ is some combination of the polynomials $p(s_i)$ which converges as $i \to \infty$. In fact, the diagonal part of $\alpha^{\mathbf{n}_i}_{\mathcal{H}}$ will multiply any entry of $U^{s\mathbf{c}}$ by some scalar whose absolute value depends exponentially on t_i, and when we multiply by the unipotent matrix $U^{t\mathbf{m}}$ we will possibly produce combinations of various entries of $U^{s\mathbf{c}}$ including a term independent of s for each matrix entry. However, note that after taking these combinations the highest degree term (as a polynomial of s) in each block still only appears in the rows corresponding to κ_0. When we choose $\mathbf{n}_i$ in the logarithmic sequence, we need to make sure that these highest order terms are made smaller than one, i.e., t_i needs to be at least so big that $|\lambda|^{t_i}|s_i|^e \leq 1$ where e is the maximal degree in the block and λ is the corresponding eigenvalue, i.e., t_i has a logarithmic lower bound. (To justify that statement one also needs to show that there cannot be cancellation between the various terms that contribute to a matrix entry. This follows also from the argument below: If $|\lambda|^{t_i}|s_i|^{e+1} \leq 1$ then all terms go to zero. So t_i also has a logarithmic upper bound and all of its powers are negligible in comparison to s.) This shows that in all other rows where the degree in s_i is smaller, we have expressions which are bounded by $|\lambda|^{t_i}|s_i|^{e-1} t_i^d \leq |\lambda|^{\frac{1}{e} t_i} t_i^d$ where d is the dimension of $V^-_{\mathcal{H}}$. Therefore, these terms go to zero along logarithmic sequences and it follows that the limit A_j along the logarithmic sequences has its values also in κ_0. ∎

4.2. E-bundles $\tilde{K}_j$ and $\tilde{L}_j$. For every κ_j in the logarithmic filtration we define $\tilde{K}_j = \mathbb{T}^d \times \kappa_j \subseteq \tilde{V}^-_{\mathcal{H}}$. Clearly this gives an e-bundle over the foliation

K_j induced by κ_j. By construction, K_j and $\tilde{K}_j$ are invariant under α and $\tilde{\alpha}$ respectively.

We say a sequence (p_i, x_i) in $\tilde{W}^-_{\mathcal{H}}$ converges to the zero section if it is bounded and all accumulation points lie in the zero section $\mathbb{T}^d \times \{0\}$. Then we define

$$\tilde{L}_j = \{(p, x) \in \tilde{W}^-_{\mathcal{H}} | \tilde{\beta}^{\mathbf{n}_i}(p, x) \text{ converges to the zero section}\}.$$

Here $\mathbf{n}_i$ is the logarithmic sequence that defined κ_j as in Section 4.1.

The following is the first of the promised matching statements; we show using the logarithmic sequences that $\tilde{\phi}$ conjugates $\tilde{K}_j$ to another e-bundle $\tilde{L}_j$.

PROPOSITION 4.4: *The e-bundle structure of $\tilde{W}^-_{\mathcal{H}}$ induces the e-bundle $\tilde{L}_j$ over the foliation $L_j = e(\tilde{L}_j) = \phi(K_j)$ for $j = 0, \ldots, \ell$. Moreover, $\tilde{\phi}(\tilde{K}_j) = \tilde{L}_j$.*

Proof: Since $\tilde{\beta}$ is linear on the fibers of $\tilde{W}^-_{\mathcal{H}}$ it follows that $\tilde{L}_j(p)$ is a vector space for every $p \in \mathbb{T}^d$, i.e., $\tilde{L}_j$ is a bundle. Moreover, induction on j shows that $\tilde{\phi}(\tilde{K}_j) = \tilde{L}_j$ — because $\tilde{L}_j$ is defined by a topological property and $\tilde{\phi}$ is a homeomorphism. To know that $\tilde{L}_j$ is an e-bundle we also need to define and check compatibility with the foliation L_j. Since $\beta^{\mathbf{n}}(e(p, x)) = e(\tilde{\beta}^{\mathbf{n}}(p, x))$ the fiber $\tilde{L}_j(p)$ is mapped to precisely those points in $L_{j+1}(p)$ whose orbits are asymptotic to the orbit of p along $\mathbf{n}_i$. Since this is again a topological characterization, it follows that $e_p(\tilde{L}_j(p)) = \phi(K_j(p))$ for $p \in \mathbb{T}^d$ — this defines the foliation L_j.

To see that $\tilde{L}_j$ is an e-bundle over L_j it remains to show that for two points $\phi(p), \phi(q) \in \mathbb{T}^d$ in the same L_j-leaf the map $\Theta_{\phi(q)\phi(p)}$ maps $\tilde{L}_j(p)$ to $\tilde{L}_j(q)$. However, $\tilde{\phi}$ is an isomorphism between $\tilde{V}^-_{\mathcal{H}}$ and $\tilde{W}^-_{\mathcal{H}}$ and since $\tilde{K}_j$ is an e-bundle, the same is true for $\tilde{L}_j$. ∎

4.3. Quotient e-bundles $\tilde{K}_{j+1}/\tilde{K}_j$ and $\tilde{L}_{j+1}/\tilde{L}_j$.

We need to define the quotient bundles $\tilde{K}_{j+1}/\tilde{K}_j$ and $\tilde{L}_{j+1}/\tilde{L}_j$, which will appear in the proof of the main result.

First note that for each $p \in \mathbb{T}^d$, $\tilde{K}_j(p) = \{p\} \times \kappa_j$ is a subspace of $\tilde{K}_{j+1}(p) = \{p\} \times \kappa_{j+1}$, so that the quotient $\tilde{K}_{j+1}(p)/\tilde{K}_j(p) = \{p\} \times (\kappa_{j+1}/\kappa_j)$ is defined. We just need to check that this quotient structure is respected by the change-of-coordinate Θ maps. For p and q in the same leaf of $\tilde{K}_{j+1}$, the lifts $\overline{p}$ and $\overline{q}$ of these points to $\mathbb{R}^d$ can be arranged to lie in the same coset of κ_{j+1}. The map $\Theta^{\tilde{K}_{j+1}}_{pq}$ maps (q, x) to $(p, x + \overline{q} - \overline{p})$. Thus $\Theta^{\tilde{K}_{j+1}}_{pq}$ maps cosets of κ_j in $\tilde{K}_{j+1}(q)$ to themselves. From this, we see that

$$\Theta^{\tilde{K}_{j+1}/\tilde{K}_j}_{pq}([x]) = [\Theta^{\tilde{K}_{j+1}}_{pq}(x)]$$

produces a well-defined map for each p and q in the same leaf of K_{j+1}. The properties that

$$\Theta_{up}^{\tilde{K}_{j+1}/\tilde{K}_j} \circ \Theta_{pq}^{\tilde{K}_{j+1}/\tilde{K}_j} = \Theta_{uq}^{\tilde{K}_{j+1}/\tilde{K}_j}$$

and that

$$\Theta_{pp}^{\tilde{K}_{j+1}/\tilde{K}_j} = Id$$

follow from the corresponding properties for the Θ maps of $\tilde{K}_{j+1}$.

Again we are able to match the above e-bundle $\tilde{K}_{j+1}/\tilde{K}_j$ with another.

Proposition 4.5: *The above defines an e-bundle $\tilde{K}_{j+1}/\tilde{K}_j$ over the foliation K_{j+1}. Similarly, there exists an e-bundle $\tilde{L}_{j+1}/\tilde{L}_j$ over the foliation L_{j+1} with fibers $\tilde{L}_{j+1}(p)/\tilde{L}_j(p)$ for every $p \in \mathbb{T}^d$. For every $(p,x) \in \tilde{K}_{j+1}$ we have*

$$\tilde{\phi}((p,x) + \tilde{K}_j(p)) = \tilde{\phi}(p,x) + \tilde{L}_j(\phi(p). \tag{4.1}$$

Proof: The cosets of $\tilde{K}_j(p)$ inside of $\tilde{K}_{j+1}(p)$ can be identified as the equivalence classes under the following equivalence relation: (p,x) and (p,y) are equivalent if the distance between $\tilde{\alpha}^{\mathbf{n}_i}(p,x)$ and $\tilde{\alpha}^{\mathbf{n}_i}(p,y)$ goes to zero when $\alpha^{\mathbf{n}_i}$ is a logarithmic sequence for the action restricted to $\tilde{K}_{j+1}$. The cosets of $\tilde{L}_j(\phi(p))$ inside of $\tilde{L}_{j+1}(\phi(p))$ can be identified in the same way. Since $\tilde{\phi}$ is a homeomorphism and the orbits in question are bounded, equation (4.1) follows. Since $\tilde{K}_{j+1}/\tilde{K}_j$ is an e-bundle and $\tilde{\phi}$ is an isomorphism, the proposition follows. ∎

5. Orbit closure arguments

The following propositions lie at the heart of our geometric method. Proposition 5.6 shows us that a conjugacy equation at a fixed point gives a very special form for the conjugacy map along certain curves. These curves arise as orbit closures from isometric directions of the higher rank linear action. Proposition 5.9 shows that from this information at a dense set of points we can deduce smoothness at every point. For this to work we will need to know that the uniform limit of a sequence of smooth functions within a certain class is again smooth — we will consider sums of products of polynomials and exponential functions and need to develop some elementary properties for that class.

In order to be able to apply these arguments inductively, as we need to build up smoothness along the filtration K_i, the propositions must be stated in enough generality to describe maps of $\tilde{K}_i$ or $\tilde{K}_{i+1}/\tilde{K}_i$. This is the reason for using the e-bundle terminology.

In what follows, we will use standard multi-index notation. For $\lambda = (\lambda_1, \lambda_2)$ a tuple of nonzero complex numbers and $\mathbf{k} = (k_1, k_2)$ a tuple of real numbers, $\lambda^{\mathbf{k}} = \lambda_1^{k_1}\lambda_2^{k_2}$.

Definition 5.1: A *polynomial sum of exponentials* (PSE) is a sum

$$p^E(\mathbf{n}) = \sum_{\lambda \in E} p^\lambda(\mathbf{n})\lambda^{\mathbf{n}},$$

where E is a finite subset of $(\mathbb{C} \setminus \{0\})^2$ and $p^\lambda(\mathbf{n}) \in \mathbb{C}[n_1, n_2]$ is a nonzero polynomial for all $\lambda \in E$. We refer to λ as the base and to $p^\lambda(\mathbf{n})$ as the polynomial coefficient (for λ).

Definition 5.2: A *PSE-function* $p^F(t)$ for some finite set F of complex numbers is a sum

$$p^F(t) = \sum_{\mu \in F} p^\mu(t)e^{\mu t},$$

where $p^\mu(t) \in \mathbb{C}[t] \setminus \{0\}$ for $\mu \in F$.

The way that polynomial sums of exponentials will naturally arise is the following.

Lemma 5.3: *Suppose $f\colon \mathbb{Z}^2 \to \mathrm{GL}(k, \mathbb{R})$ is a homomorphism. Then in any basis the entries of the matrix $f(\mathbf{n})$ are PSE's.*

Proof: Consider a basis in which the generators $A = f(\mathbf{e}_0)$ and $B = f(\mathbf{e}_1)$ are simultaneously upper-triangular. Then $A^n = \mathrm{diag}(A)^n(\mathrm{diag}(A)^{-1}A)^n$. The unipotent matrix $\mathrm{diag}(A)^{-1}A$ raised the power n has entries which are polynomial in n, while the entries of $\mathrm{diag}(A)^n$ are λ^n for the different eigenvalues λ. Thus the entries of A^n are PSE's, as are those of B^m. The product of these has entries which are PSE's in $\mathbf{n} = (n, m)$. Conjugating by fixed change-of-basis matrix, the result still has entries which are PSE's. ∎

Note that the difference operator $\Delta_{\mathbf{k},\lambda}$ defined

$$\Delta_{\mathbf{k},\lambda}(p^E)(\mathbf{n}) = \frac{p^E(\mathbf{n}+\mathbf{k})}{\lambda^{\mathbf{k}}} - p^E(\mathbf{n})$$

maps a PSE to a PSE (where we allow the zero function). Similarly, the difference operator

$$\Delta_{s,\mu}(p^F)(t) = \frac{p^F(t+s)}{e^{\mu s}} - p^F(t)$$

maps a PSE-function to a PSE-function.

We will call the sum of the total degrees of p^λ for $\lambda \in E$ plus the cardinality of E the degree of $p^E(\mathbf{n})$.

Lemma 5.4: *The degree of a PSE $p^E(\mathbf{n})$ is the smallest number d such that there exist $\lambda_1, \ldots, \lambda_d$ so that for all $\mathbf{k}_1, \ldots, \mathbf{k}_d$*

$$\Delta_{\mathbf{k}_1,\lambda_1} \circ \cdots \circ \Delta_{\mathbf{k}_d,\lambda_d}(p^E) = 0. \tag{5.1}$$

Similarly, the degree of a PSE-function $p^F(t)$ is the smallest number d such that there exist $\mu_1, \ldots, \mu_d$ so that for all $s_1, \ldots, s_d$

$$\Delta_{s_1,\mu_1} \circ \cdots \circ \Delta_{s_d,\mu_d}(p^F) = 0. \tag{5.2}$$

Furthermore, if a function $f\colon \mathbb{R} \to \mathbb{C}$ satisfies

$$\Delta_{s_1,\mu_1'} \circ \cdots \circ \Delta_{s_d,\mu_d'}(f) = 0, \tag{5.3}$$

for every $s_1, \ldots, s_d \in \mathbb{R}$ and some $\mu_1', \ldots, \mu_d'$, then $f = p^F$ is a PSE-function. Here μ_i' is allowed to depend on s_i for $i = 1, \ldots, d$, and the numbers μ_i' for $i = 1, \ldots, d$ determine the real parts of the elements of F, i.e., if $F = \{\mu_1, \ldots, \mu_\ell\}$, then $\{\mathrm{Re}(\mu_1), \ldots, \mathrm{Re}(\mu_\ell)\} \subset \{\mathrm{Re}(\mu_1'), \ldots, \mathrm{Re}(\mu_d')\}$.

Proof: Consider a term $p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$ in the sum for $p^E(\mathbf{n})$. Note that

$$\Delta_{\mathbf{k},\lambda}(p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}) = \frac{p^\lambda(\mathbf{n}+\mathbf{k})\lambda^{\mathbf{n}+\mathbf{k}}}{\lambda^{\mathbf{k}}} - p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$$

either decreases the total degree of p^λ by one or sends that term to zero (decreasing the cardinality of E by one), depending on whether or not p^λ has $\mathbf{k}$-degree greater than zero. For $\lambda_0 \neq \lambda$, $\Delta_{\mathbf{k},\lambda}(p^{\lambda_0}(\mathbf{n})\lambda_0^{\mathbf{n}})$ has the same degree as $p^{\lambda_0}(\mathbf{n})\lambda_0^{\mathbf{n}}$. Thus the elements of E listed with multiplicity (according to the total degree of p^{λ_i} plus one) give a minimal list. The PSE-function case is similar.

For the second statement, consider first the case $d = 1$. Then $\Delta_{1,\mu} f = 0$. Then $f(t+n) = f(t)e^{\mu n}$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. By dividing by $e^{\mu t}$ we can assume that $\mu = 0$. This is equivalent to $f(t)$ being periodic, i.e., $f(t+n) = f(t)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Now let s_1 be irrational; then by the assumption of the lemma there exists some μ_1' such that $\Delta_{s_1,\mu_1'}(f) = 0$. Therefore, $f(t+n+ms_1) = f(t)e^{\mu_1' m}$ for all $t \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. If $f(0) = 0$ then by continuity of f and density of $\mathbb{Z} + \mathbb{Z}s_1$ we get $f = 0$. So we can assume without loss of generality that $f(0) = 1$. Therefore for all $n, m \in \mathbb{Z}$,

$$f(t+n+ms_1) = f(t+n)e^{\mu_1' m} = f(t)f(ms_1) = f(t)f(n+ms_1).$$

Since $\mathbb{Z} + \mathbb{Z}s_1$ is dense in $\mathbb{R}$ and f is continuous, this shows that $f\colon \mathbb{R} \to \mathbb{C}$ is a periodic continuous character from $\mathbb{R}$ to $\{z \in \mathbb{C} : |z| = 1\}$, i.e., $f(t) = e^{\mu_1 t}$ for all $t \in \mathbb{R}$ and some fixed $\mu_1 \in 2\pi\mathbb{Z}$. This shows the lemma in this case.

Suppose now f satisfies (5.3) with $d > 1$. By induction $\Delta_{1,\mu}(f) = p^{F'}$ for some PSE-function $p^{F'}$ (and the real parts of the elements of F' are determined). Then one easily finds a PSE-function $q^{F'}$ with $\Delta_{1,\mu}q^{F'} = p^{F'}$ and so $\Delta_{1,\mu}(f - q^{F'}) = 0$. As before we can assume $\mu = 0$ and work with periodic functions. We claim that $g(t) = (f - q^{F'})(t)$ is a sum of characters, and in particular a PSE-function. This will imply the lemma. Now let s_1 be irrational; then by the assumption of the lemma and the inductive assumption there exists some μ_1' such that $\Delta_{s_1,\mu_1'}(f)$ is a PSE-function. This shows that $\Delta_{s_1,\mu_1'}(g) = h$ is a periodic PSE-function, i.e., all polynomial coefficients of h are constants, and h is a sum of periodic characters. There exists a PSE-function h_0 with

$$\begin{aligned} \Delta_{s_1,\mu_1'}(h_0) &= h, \\ g(t+s_1) - g(t)e^{\mu_1'} &= h_0(t+s_1) - h_0(t)e^{\mu_1'} \quad \text{for all } t \in \mathbb{R}, \\ g(t+ns_1) - g(t)e^{\mu_1' n} &= h_0(t+ns_1) - h_0(t)e^{\mu_1' n} \quad \text{for all } t \in \mathbb{R}, n \in \mathbb{Z}. \end{aligned}$$

Due to periodicity, g and therefore also h_0 are bounded. This shows that the polynomial coefficients of h_0 are constants, and therefore h_0 is a sum of multiples of periodic characters (with the same characters as for h). Now we can consider $g' = g - h_0 = f - q^{F'} - h_0$ which satisfies $\Delta_{1,0}(g') = \Delta_{s_1,\mu_1'}(g') = 0$. However, we have already shown above that this implies that g' is itself also a multiple of a periodic character. Thus $g = g' - h_0$ is a PSE-function. ∎

Suppose $S \subset \mathbb{R}^d$ is a subspace. Say that S is a rational subspace if S is spanned by vectors which have rational coordinates in the standard basis of $\mathbb{R}^d$.

Definition 5.5: Suppose $f\colon \mathbb{Z}^2 \to G$ is a homomorphism to a Lie group G. We say that f is isometric on a subspace S if $\{f(\mathbf{n}) \mid \|\mathbf{n} - S\| \leq 1\}$ has compact closure.

If $G = \mathrm{GL}(k,\mathbb{R})$, this is equivalent to saying that the magnitude of all eigenvalues of $f(\mathbf{n}_i)$ goes to one whenever $\mathbf{n}_i \in \mathbb{Z}^2$ satisfy $\|\mathbf{n}_i - S\| \to 0$ and there is no shearing along S.

PROPOSITION 5.6: *Suppose $f\colon \mathbb{Z}^2 \to \mathrm{GL}(k,\mathbb{R})$ is isometric along the subspace $S \subset \mathbb{R}^2$. Suppose there is some $\mathbf{m}_0 \in \mathbb{Z}^2$ such that all eigenvalues of $f(\mathbf{m}_0)$ have norm less than one, and suppose no eigenvalues of $f(\mathbf{n})$ are roots of unity except for $\mathbf{n} = 0$. Furthermore, let $\psi\colon \mathbb{R}^k \to \mathbb{R}$ be continuous and satisfy*

$$\psi(f(\mathbf{n})\mathbf{x}) = p^E_{\mathbf{x}}(\mathbf{n})$$

for some PSE $p^E_{\mathbf{x}}$. Then there exists an invertible matrix $M \in \mathfrak{gl}(k,\mathbb{R})$ and a

PSE-function $\bar{p}^F_{\mathbf{x}}(t)$ *such that*

$$\psi(\exp(tM)\mathbf{x}) = \bar{p}^F_{\mathbf{x}}(t) \quad \text{for all } t \in \mathbb{R}. \tag{5.4}$$

The matrix M *only depends on the homomorphism* f *restricted to any finite index subgroup, and furthermore a uniform bound on the absolute values* $|\log \lambda_1|, |\log \lambda_2|$ *for all* $\lambda \in E$ *gives a bound on* $|\operatorname{Re}(\mu)|$ *for* $\mu \in F$.

Lemma 5.7: *If a PSE* $p^E(\mathbf{n})$ *is bounded along a subspace* $S \subset \mathbb{R}^2$, *i.e.,*

$$\sup_{\mathbf{n}:\operatorname{dist}(\mathbf{n},S)<r} |p^E(\mathbf{n})| < \infty \quad \text{for all } r > 0,$$

then for $\lambda \in E$ *with* $p^\lambda(\mathbf{n}) \neq 0$ *we have*

$$(\log|\lambda_1|, \log|\lambda_2|) \perp S. \tag{5.5}$$

Moreover, if we write $\mathbf{n} = s\mathbf{b} + t\mathbf{m}_0$ *for some fixed* $\mathbf{b} \in S - \{0\}$ *and* $\mathbf{m}_0 \notin S$, *then* $p^\lambda(s\mathbf{b} + t\mathbf{m}_0)$ *only depends on* t.

Proof: For a function of the form

$$p^E(\mathbf{n}) = p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$$

this is trivial. For a more general PSE it follows by applying the difference operator $\Delta_{\mathbf{k},\lambda}$ and the observation that $\Delta_{\mathbf{k},\lambda}(p^E)$ is bounded along S if p^E is. ∎

Proof of Proposition 5.6: Let H be the closure of $f(\mathbb{Z}^2)$ in $\mathrm{GL}(k,\mathbb{R})$, and let $\mathfrak{h}$ be its Lie algebra. We claim there exists an invertible $M \in \mathfrak{h}$.

Suppose S is irrational. Since f is isometric along S and every translate of S has lattice points arbitrarily close to it, it follows that $\mathfrak{h}$ is nontrivial. Let $M \in \mathfrak{h} - \{0\}$ be such that $\exp(M) = f(\mathbf{m}_1)$ with $\mathbf{m}_1$ in the same halfspace as the $\mathbf{m}_0$ in the assumption. Then since no vector in $\mathbb{R}^k$ can be fixed under $f(\mathbf{m}_1) = \exp(M)$ by our assumptions on f, it follows that M is invertible. Similarly, if S is rational and $\mathbf{b} \in S \cap \mathbb{Z}^k - \{0\}$ then $f(\mathbf{b}\mathbb{Z})$ is an infinite subgroup of a compact group and so $\mathfrak{h}$ is nontrivial. Let $M \in \mathfrak{h}$ be such that $\exp(M) = f(\mathbf{b}_1)$ for some $\mathbf{b}_1 \in S \cap \mathbb{Z}^2 \setminus \{0\}$. Since no root of unity is an eigenvalue of $f(\mathbf{b}_1)$, zero cannot be an eigenvalue of M, so M is invertible.

If we had the same homomorphism f restricted to a finite index subgroup of $\mathbb{Z}^2$, the Lie algebra $\mathfrak{h}$ would be the same. Thus we could choose the same M in that case.

The cases of rational and irrational S are slightly different with the rational case being slightly easier: If S is rational, we can work just with the subaction corresponding to $\mathbb{Z}\mathbf{b}$ and for these values of $\mathbf{n}$ the polynomial coefficients are just constants. Also, in this case a bound on $\mathrm{Re}(\mu)$ is trivial since μ will be obtained from powers of $\lambda^{\mathbf{b}}$ and will be purely imaginary. From now on we consider only the case where S is irrational.

Let $\mathbf{x} \in \mathbb{R}^k$ be fixed and consider the PSE $p_{\mathbf{x}}^E(\mathbf{n})$. By Lemma 5.7, the polynomial coefficients only depend on the distance to S and the bases appearing satisfy (5.5). We continue as in the proof of Lemma 5.7, considering first the case of a PSE of the form $p^E(\mathbf{n}) = p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$. Choose for $t \in \mathbb{R}$ a sequence $\mathbf{n}_i \in \mathbb{Z}^2$ with $f(\mathbf{n}_i) \to \exp(tM)$. Then the distance of $\mathbf{n}_i$ to S converges as well; for an appropriate rescaling of the metric the limit is t. Let $\bar{p}^\lambda(t) = p^\lambda(t\mathbf{m}_0)$. By continuity of ψ, from $\psi(f(\mathbf{n})\mathbf{x}) = p^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$ we get

$$p^\lambda(\mathbf{n}_i)\lambda^{\mathbf{n}_i} \to \bar{p}^\lambda(t)\phi^\lambda(t) = \psi(\exp(tM)\mathbf{x})$$

as $i \to \infty$. If $\bar{p}^\lambda(t) \neq 0$ this determines $\phi^\lambda(t)$ uniquely. It is easy to see that $\phi^\lambda\colon \mathbb{R} \to \mathbb{C}$ is a homomorphism which is continuous since ψ is. Thus in this special case $\phi^\lambda(t) = e^{\mu t}$ as desired, showing that equation (5.4) holds. We also see that $|e^\mu|$ is determined by the size of $(\log|\lambda_1|, \log|\lambda_2|)$.

Now suppose that $p_{\mathbf{x}}^E(\mathbf{n}) = \sum_{\lambda\in E} p_{\mathbf{x}}^\lambda(\mathbf{n})\lambda^{\mathbf{n}}$ with E containing more than one point. For each λ_0 in E we need to show there is some μ_0 such that $p^{\lambda_0}(\mathbf{n}_i)\lambda_0^{\mathbf{n}_i}$ converges to $\bar{p}^{\lambda_0}(t)e^{\mu_0 t}$ when $f(\mathbf{n}_i)$ converges to $\exp(tM)$. To do this, fix $\lambda_0 \in E$ and consider another $\lambda \in E$ $(\lambda_0 \neq \lambda)$. Choose some $\mathbf{k} \in \mathbb{Z}^2$ with $\lambda^{\mathbf{k}} \neq \lambda_0^{\mathbf{k}}$, and consider the function

$$\bar{\psi}(\mathbf{y}) = \frac{\psi(f(\mathbf{k})\mathbf{y})}{\lambda^{\mathbf{k}}} - \psi(\mathbf{y}),$$

which satisfies

$$\bar{\psi}(f(\mathbf{n})\mathbf{x}) = \Delta_{\mathbf{k},\lambda} p_{\mathbf{x}}^E(\mathbf{n}) \quad \text{for } \mathbf{n} \in \mathbb{Z}^2.$$

This $\Delta_{\mathbf{k},\lambda} p_{\mathbf{x}}^E(\mathbf{n})$ is a PSE which has a λ_0 term of the same degree as $p_{\mathbf{x}}^E$ and a λ term, if present, of lower degree. By repeatedly making such substitutions for all λ in E, we eventually reach the simpler case above where only a λ_0 term remains. We conclude that $\lambda_0^{\mathbf{n}_i} \to e^{\mu_0 t}$ whenever $f(\mathbf{n}_i) \to \exp(tM)$. Since we can do this for each $\lambda_0 \in E$, the proposition follows. ∎

Lemma 5.8: *Suppose that $p_\ell^{F_\ell}(t)$ is a sequence of PSE-functions where the degree of $p_\ell^{F_\ell}(t)$ and $\mathrm{Re}(\mu)$ for $\mu \in F_\ell$ are uniformly bounded. If $p_\ell^{F_\ell}(t) \to q(t)$ uniformly on compact sets as $\ell \to \infty$, then $q(t)$ is also a PSE-function $q^F(t)$.*

Proof: By choosing a subsequence we can assume that each of the PSE-functions has degree d. Let $s_1, \ldots, s_d \in \mathbb{R}$. Listing the elements of F_ℓ with multiplicities (according to the degrees) we have the elements $\mu_\ell^1, \ldots, \mu_\ell^d \in F_\ell$. Choosing a subsequence we can assume that $e^{\mu_\ell^i s_i} \to e^{\mu^i s_i}$ as $\ell \to \infty$ for $i = 1, \ldots, k$. Note that $\Delta_{s_i, \mu_\ell^i} p_\ell^{F_\ell} \to \Delta_{s_i, \mu^i} q$ uniformly on compact sets as $\ell \to \infty$. This shows that

$$\Delta_{s_1, \mu^1} \circ \cdots \circ \Delta_{s_d, \mu^d}(q) = 0.$$

Now the lemma follows from Lemma 5.4. ∎

Next we use the above orbit closure statements to show smoothness for e-bundle homomorphisms.

PROPOSITION 5.9: *Suppose that $\tilde{K}$ is either the e-bundle $\tilde{K}_{i+1}$ or the $\tilde{K}_{j+1}/\tilde{K}_j$. In either case this is an e-bundle over the foliation $K = K_{j+1}$. Suppose $\tilde{L}$ is another e-bundle over a foliation L of $\mathbb{T}^d$, for which $\tilde{\psi}$ is an e-bundle map from $\tilde{K}$ to $\tilde{L}$. Let k and k' be the dimensions of the fibers of $\tilde{K}$ and $\tilde{L}$, respectively. Suppose that there is a dense set P of points in $\mathbb{T}^d$ such that for $p \in P$,*

$$\tilde{\psi}((p, \exp(tM)\mathbf{x})) = \big(\psi(p), p_{\mathbf{x}}(t)\big) \tag{5.6}$$

holds. Assume that in (5.6), M is an invertible k by k matrix (independent of p and $\mathbf{x}$) and $p_{\mathbf{x}}$ is a vector valued function whose components are PSE-functions (depending on p and $\mathbf{x}$). Suppose the degrees of the PSE-functions and the real parts $\mathrm{Re}(\mu)$ of their exponents $\mu \in F_x$ are uniformly bounded. Then $\tilde{\psi}$ is a smooth e-bundle map.

In the proof we will make use of the following regularity theorem.

THEOREM 5.10 ([Rud91, Thm. 8.12]): *Let Ω be an open set in $\mathbb{R}^d$ and D an elliptic linear partial differential operator of order k with constant coefficients. Let u be a distribution on Ω and $Du = f$ with f locally in the Sobolev space H^s. Then u is locally H^{s+k}.*

Proof: The change-of-coordinate maps $\Theta_{pq}^{\tilde{K}}$ for the e-bundle $\tilde{K}$ are given by

$$\Theta_{pq}^{\tilde{K}}([x]) = [\Theta_{pq} x] = [x + q - p],$$

where we identify p, q with their lifts to $\mathbb{R}^d$ and assume that $q - p \in \kappa_{j+1}$. Also note that in the case $K = \tilde{K}_{j+1}$ the equivalence class of x is just x, while in the case $K = \tilde{K}_{j+1}/\tilde{K}_j$ the equivalence class of x is $x + \kappa_j$.

Let $q \in \mathbb{T}^d$ be any point. We want to show that the restriction of $\tilde{\psi}$ to the leaf $\tilde{K}(q)$ is smooth on a neighborhood of 0. Since the coordinate changes $\Theta_{pq}^{\tilde{K}}$ can map any point in the fiber to the zero-section (at the point p) this is sufficient.

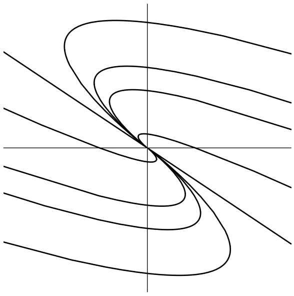

Figure 5. Curves $\exp(tM)\mathbf{x}$ foliate $\tilde{K}(p) - \{0\}$.

Note that for each $p \in P$ the curves $\exp(tM)\mathbf{x}$ for $\mathbf{x} \in \tilde{K}(p)$ (as in (5.6)) foliate $\tilde{K}(p) - \{0\}$ (Figure 5). We will show that we get a similar functional equation to (5.6) for a limit point of a sequence $p_i \subset P$ — in particular, smoothness along the curves holds also at the limit point. For a limit point q' in the leaf $K(q)$, the map $\Theta_{qq'}$ is defined and translates this foliation. By choosing k different translations of this foliation, we can cover a neighborhood U of 0 in $\tilde{K}(q)$ by a full transverse set of foliations. Indeed, note first that the curve $\exp(tM)\mathbf{x}$ is tangent at $t = 0$ to $M\mathbf{x}$. Then choose points $q_1, q_2, \ldots, q_k \in K(q)$ so that $q_1 - q, \ldots, q_k - q$ are linearly independent in κ_{i+1} respectively κ_{i+1}/κ_i. Since M is invertible, we get that $M(q_1 - q), M(q_2 - q), \ldots, M(q_k - q)$ are linearly independent. Therefore, the translated foliations for these k points give a full transverse set of foliations near $0 \in \tilde{K}(q)$. Also note that these foliations are quite concrete foliations defined by the matrix M. We will show below that $\tilde{\psi}$ is smooth along every leaf of every one of these foliations. These foliations define a smooth local coordinate system on the neighborhood $U \subset \tilde{K}(q)$ of 0, say $z_1, \ldots, z_k$ are the coordinates, such that each of the foliations is the foliation along the curves where only one coordinate changes. Therefore, for all integers $\ell > 0$,

$$D = \frac{\partial^{2\ell}}{\partial z_1^{2\ell}} + \cdots + \frac{\partial^{2\ell}}{\partial z_k^{2\ell}}$$

is an elliptic differential operator such that $D(\tilde{\psi}|_U)$ is continuous. By Theorem 5.10 we see that $\tilde{\psi} \in H^{2\ell}$ for all ℓ. Since the intersection of these Sobolev spaces is C^∞ the proposition will follow.

It remains to show that $\tilde{\psi}$ is smooth along a translate of the foliation given by the curves $\exp(tM)\mathbf{x}$ for $x \in \tilde{K}(p)$ with the derivatives being continuous on $\tilde{K}(p)$.

Fix some point $q' \in K(q)$ and let $(p_i)_{i=1}^{\infty}$ be a sequence of points in P which converge to q' as $i \to \infty$. Furthermore, choose some $q_i \in K(p_i)$ such that $p_i - q_i \to q' - q$ for $i \to \infty$. By assumption the e-bundle structure of $\tilde{K}$ coincides with the linear structure of the torus. This means that the maps $\mathbf{y} \mapsto \mathbf{y} + p_i - q_i$ from $\tilde{K}(p_i)$ to $\tilde{K}(q_i)$ are the change-of-coordinate maps for the e-bundle structure. The assumption that $\tilde{\psi}$ is an e-bundle map is an assumption that there are corresponding change-of-coordinate maps $\Theta_{\psi(q_i)\psi(p_i)}$ for the e-bundle F with

$$\tilde{\psi}(\mathbf{x} + p_i - q_i) = \Theta_{\psi(q_i)\psi(p_i)} \circ \tilde{\psi}(\mathbf{x}).$$

Thus

$$\begin{aligned}\tilde{\psi}_{q_i}(\exp(tM)\mathbf{x} + p_i - q_i) &= \Theta_{\psi(q_i)\psi(p_i)} \circ \tilde{\psi}(\exp(tM)\mathbf{x}) \\ &= \Theta_{\psi(q_i)\psi(p_i)}(p_{\mathbf{x},i}(t)).\end{aligned}$$

As $j \to \infty$ the left side converges to $\tilde{\psi}_q(\exp(tM)\mathbf{x} + q' - q)$, which is precisely $\tilde{\psi}_q$ restricted to one of the curves in the translated foliation. By Proposition 3.7 the function e is continuous, which by Corollary 3.8 shows that $\Theta_{\psi(q_j)\psi(p_j)}$ converges to $\Theta_{\psi(q)\psi(q')}$. Note that $p_{\mathbf{x},j}(t)$ depends on j in some uncontrolled fashion. However, we know that $p_{\mathbf{x},j}(t)$ converges uniformly on compact sets to a function $q_{\mathbf{x}}$. By Lemma 5.8 it follows that $q_{\mathbf{x}}$ is a vector valued function whose components are PSE-functions. We get

$$\tilde{\psi}_q(\exp(tM)\mathbf{x} + q' - q) = \Theta_{\psi(q)\psi(q')}(q_{\mathbf{x}}(t)).$$

This shows smoothness. By using the chain rule and smoothness of $\Theta_{\psi(q)\psi(q')}$ we see furthermore that all derivatives with respect to t are continuous functions of $\mathbf{x}$. ∎

6. Proof of Theorem 1.1

We show first by induction that $\tilde{\phi}$ restricted to $\tilde{K}_j$ is a linear e-bundle morphism.

6.1. Start of Induction. The base case is $\tilde{K}_0$. Suppose p is periodic under α. Pass to a sub-$\mathbb{Z}^2$ action α_p under which p is fixed. Let β_p be the corresponding subaction of β. Note that by the construction of the filtration, there is no shear in κ_0 along the direction of $\partial\mathcal{H}$. Therefore, α_p restricted to the fibers of $\tilde{K}_0$ acts isometrically along the critical direction $\partial\mathcal{H}$. By Lemma 5.3, the entries of $\tilde{\beta}_p^{\mathbf{n}}$ are PSE's, so the conjugacy

$$\tilde{\phi}(\tilde{\alpha}_p^{\mathbf{n}}\mathbf{x}) = \tilde{\beta}_p^{\mathbf{n}}(\tilde{\phi}(\mathbf{x})),$$

when restricted to the fiber $\tilde{K}_0(p)$, has on the right side a vector whose entries are PSE's. Thus Proposition 5.6 applies. This gives us

$$\tilde{\psi}((p, \exp(tM)\mathbf{x})) = (\psi(p), p_{\mathbf{x}}(t)),$$

where $p_{\mathbf{x}}(t)$ are vectors-valued PSE-functions for each $\mathbf{x}$. We can do this at an arbitrary periodic point p, and periodic points for the action are dense. Moreover, since the action α_p is just the restriction of α to a finite index subgroup of $\mathbb{Z}^2$, Proposition 5.6 also shows that M can be chosen independently of p. (In the proof of this result M was found using the Lie algebra of the closure of the group defined by α.) Finally, Proposition 5.9 applies giving smoothness of $\tilde{\phi}$ in the fibers of $\tilde{K}_0$.

LEMMA 6.1: *Assume α, β are as in Theorem 1.1. Let K and L be bundles over $\mathbb{T}^d$ such that α and β have natural linear extensions $\tilde{\alpha}$ on K and $\tilde{\beta}$ on L. Assume there is some $\mathbf{n}$ such that $\tilde{\alpha}^{\mathbf{n}}$ and $\tilde{\beta}^{\mathbf{n}}$ are contracting in the fibers, have narrow spectrum, and that the two spectra are close. Suppose ψ: $K \to L$ conjugates $\tilde{\alpha}$ and $\tilde{\beta}$, preserves the zero section, and is smooth in the fibers. Then ψ is linear in the fibers.*

Proof: Let $p \in \mathbb{T}^d$ be a periodic point for α and assume $\mathbf{n}$ such that $\alpha^{\mathbf{n}}(p) = p$. (We might have to take a multiple of the $\mathbf{n}$ above.) Let $(q, 0) = \psi(p, 0)$. We will show that ψ_p: $K(p) \to L(q)$ is linear. Write a linear expansion for ψ_p:

$$\psi_p(v) = Av + r(v),$$

where the remainder term satisfies $\|r(v)\| \le c\|v\|^2$ for some constant c. The conjugacy equation gives

$$\psi_p(v) = \tilde{\beta}^{-i\mathbf{n}}\psi_p(\tilde{\alpha}^{i\mathbf{n}}v) = \tilde{\beta}^{-i\mathbf{n}}A\tilde{\alpha}^{i\mathbf{n}}v + \tilde{\beta}^{-i\mathbf{n}}r(\tilde{\alpha}^{i\mathbf{n}}v).$$

Note that for fixed i, $\|\tilde{\beta}^{-i\mathbf{n}}r(\tilde{\alpha}^{i\mathbf{n}}v)\| \le c'\|v\|^2$ for some constant $c' = c'(i)$ as $\tilde{\beta}^{-i\mathbf{n}}$ and $\tilde{\alpha}^{i\mathbf{n}}$ are just linear maps. Then by the uniqueness of linear expansions for ψ_p, we must have $r(v) = \tilde{\beta}^{-i\mathbf{n}}r(\tilde{\alpha}^{i\mathbf{n}}v)$.

Since we have assumed that α and β have narrow spectrum, the same holds for the linear maps $\tilde{\alpha}_p^{\mathbf{n}}$: $K(p) \to K(p)$ and $\tilde{\beta}_q^{\mathbf{n}}$: $L(q) \to L(q)$. (Narrow spectrum is preserved under taking powers.) Therefore we can choose λ larger than the spectral radius of $\tilde{\alpha}_p^{\mathbf{n}}$ and μ larger than the spectral radius of $\tilde{\beta}_q^{-\mathbf{n}}$ with $\lambda^2\mu < 1$.

For i large enough $\|\tilde{\beta}^{-i\mathbf{n}}\| < \mu^i$ and $\|\tilde{\alpha}^{i\mathbf{n}}\| < \lambda^i$. Then

$$r(v) = \|\tilde{\beta}^{-i\mathbf{n}} r(\tilde{\alpha}^{i\mathbf{n}} v)\| \leq c\mu^i \|\tilde{\alpha}^{i\mathbf{n}} v\|^2 \leq c\mu^i \lambda^{2i} \|v\|^2,$$

which converges to 0 as i goes to ∞. Thus $r(v) = 0$, i.e., ψ is linear in the fiber $K(p)$. But since this holds at every periodic point p, ψ is linear on a dense set of fibers. Then by continuity, in fact ψ is linear on every fiber. ∎

This lemma applies to show that $\tilde{\phi}$ is in fact linear in the fibers of $\tilde{K}_0$, completing the base case of the induction.

6.2. Inductive Step. There are two portions of the inductive step. The first of these is to show that the conjugacy from the quotient bundle $\tilde{K}_{j+1}/\tilde{K}_j$ to $\tilde{L}_{j+1}/\tilde{L}_j$ is linear.

Note that by Lemma 4.1 the action induced by α on the linear space κ_{j+1}/κ_j has no shear along $\partial\mathcal{H}$, since it is conjugated to the restriction of α to a subspace of κ_0. Therefore, the action $\tilde{\alpha}$ on $\tilde{K}_{j+1}/\tilde{K}_j$ also has no shear along $\partial\mathcal{H}$. This action is conjugated to the action induced by β on $\tilde{L}_{j+1}/\tilde{L}_j$, producing a situation completely analogous to the base case of the induction. As before we apply Proposition 5.6, Proposition 5.9, and Lemma 6.1 to conclude that the conjugation is linear.

The more difficult portion is to show linearity for the shearing terms of $\tilde{\phi}$ from K_{j+1} into K_j. Let p be a periodic point for the α action, and pass to a sub-$\mathbb{Z}^2$ action α_p that fixes p. Let β_p be the corresponding subaction of β that fixes $\phi(p)$. We choose a basis of $\tilde{K}_j(p)$ and extend it to a basis of $\tilde{K}_{j+1}(p)$. By assumption we know that $\tilde{\phi}$: $\tilde{K}_j(p) \to \tilde{L}_j(p)$ is linear and by the above discussion we know that the map induced by ϕ from $\tilde{K}_{j+1}(p)/\tilde{K}_j(p)$ to $\tilde{L}_{j+1}(p)/\tilde{L}_j(p)$ is linear. Moreover, $\tilde{\phi}$ is bijective. Therefore, we can take the image of the chosen basis of $\tilde{K}_{j+1}(p)$ as the basis of $\tilde{L}_{j+1}(p)$. (Of course we do not have any a priori smoothness properties of this basis as a function of the base point p.)

We write $\mathbf{x}$ for the coordinates corresponding to the basis elements of $\tilde{K}_j(p)$ and $\mathbf{y}$ for the remaining coordinates. Write $\tilde{\phi}_p$ for the restriction of ϕ to $\tilde{K}_{j+1}(p)$. Then

$$\tilde{\phi}_p \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}, \mathbf{y}) \\ \mathbf{y} \end{pmatrix}, \tag{6.1}$$

where the $f(\mathbf{x}, \mathbf{y})$ gives the coordinates corresponding to the basis of $\tilde{L}_j(p)$. Let

$$\tilde{\alpha}_p^{\mathbf{n}} = \begin{pmatrix} \tilde{\alpha}_{00}^{\mathbf{n}} & \tilde{\alpha}_{01}^{\mathbf{n}} \\ 0 & \tilde{\alpha}_{11}^{\mathbf{n}} \end{pmatrix} \quad \text{and} \quad \tilde{\beta}_p^{\mathbf{n}} = \begin{pmatrix} \tilde{\beta}_{00}^{\mathbf{n}} & \tilde{\beta}_{01}^{\mathbf{n}} \\ 0 & \tilde{\beta}_{11}^{\mathbf{n}} \end{pmatrix}$$

be block matrices in the $\binom{\mathbf{x}}{\mathbf{y}}$ coordinates. Note that by the construction $\tilde{\alpha}_{11}$ is isometric along the critical direction $\partial\mathcal{H}$.

LEMMA 6.2: *The function $f(\mathbf{x},\mathbf{y})$ is an affine function of $\mathbf{x}$ for a fixed $\mathbf{y}$.*

Proof: First note that $f_{\mathbf{y}}(\mathbf{x})$ is smooth in $\mathbf{x}$ and, moreover, all derivatives with respect to $\mathbf{x}$ are still continuous functions of $\mathbf{x}$ and $\mathbf{y}$. The smoothness can immediately be seen by writing

$$(f_{\mathbf{y}}(\mathbf{x}),\mathbf{y}) = \tilde{\phi}(p,(\mathbf{x},\mathbf{y})) = \Theta_{\phi(p)\phi(q)} \circ \tilde{\phi}(q,(\mathbf{x},0)),$$

where $q = p + (0,\mathbf{y})$. Then $\tilde{\phi}(q,(\mathbf{x},0)) \in \tilde{L}_j(\phi(q))$ depends linearly on $\mathbf{x}$. Composing with $\Theta_{\phi(p)\phi(q)}$ and applying the chain rule shows that the derivatives of $f_{\mathbf{y}}(\mathbf{x})$ are linear combinations of the partial derivatives of $\Theta_{\phi(p)\phi(q)}$. The coefficients of this linear combination are the coefficients of the linear function $\tilde{\phi}(q,(\mathbf{x},0))$. Since $\tilde{\phi}$ is continuous, the claim follows.

Now fix a particular $(\mathbf{x},\mathbf{y})$ and a contraction $\tilde{\alpha}^{\mathbf{n}}$. We will show that the error term to the linear approximation for $f_{\mathbf{y}}(\mathbf{x})$ (based at $\mathbf{x}_0 = 0$) is zero. Write the linear approximation as

$$f_{\mathbf{y}}(\mathbf{x}) = a_0 + b_0(\mathbf{x}) + r_0(\mathbf{x}),$$

where the remainder satisfies $\|r_0(\mathbf{x})\| < c\|\mathbf{x}\|^2$ for a fixed c whenever $(\mathbf{x},\mathbf{y})$ belong to a compact set. This uniform estimate on the remainder follows from the continuity of the derivatives observed above. We will make use of the linear approximation for $f_{\tilde{\alpha}_{11}^{i\mathbf{n}}\mathbf{y}}(\mathbf{z})$ with base point $\tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y}$, which we write as

$$f_{\tilde{\alpha}_{11}^{i\mathbf{n}}\mathbf{y}}(\mathbf{z}) = a_i + b_i(\mathbf{z} - \tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y}) + r_i(\mathbf{z} - \tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y}).$$

Again there is the estimate $\|r_i(\mathbf{z} - \tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y})\| < c\|\mathbf{z} - \tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y}\|^2$ whenever $(\mathbf{x},\mathbf{y})$ belong to the same compact set as above.

The conjugacy equation $\tilde{\phi} = \tilde{\beta}_p^{-\mathbf{n}} \circ \tilde{\phi} \circ \tilde{\alpha}_p^{\mathbf{n}}$ gives us

$$\begin{aligned}
\begin{pmatrix} f_{\mathbf{y}}(\mathbf{x}) \\ \mathbf{y} \end{pmatrix} &= \tilde{\beta}_p^{-i\mathbf{n}} \tilde{\phi} \begin{pmatrix} \tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x} + \tilde{\alpha}_{01}^{i\mathbf{n}}\mathbf{y} \\ \tilde{\alpha}_{11}^{i\mathbf{n}}\mathbf{y} \end{pmatrix} \\
&= \tilde{\beta}_p^{-i\mathbf{n}} \begin{pmatrix} a_i + b_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x}) + r_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x}) \\ \tilde{\alpha}_{11}^{i\mathbf{n}}\mathbf{y} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\beta}_{00}^{-i\mathbf{n}} a_i + \tilde{\beta}_{01}^{-i\mathbf{n}} \tilde{\alpha}_{11}^{i\mathbf{n}}\mathbf{y} + \tilde{\beta}_{00}^{-i\mathbf{n}} b_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x}) + \tilde{\beta}_{00}^{-i\mathbf{n}} r_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x}) \\ \mathbf{y} \end{pmatrix}.
\end{aligned}$$

Note that the final expression gives a linear expansion for $f_{\mathbf{y}}(\mathbf{x})$ at base point $\mathbf{x}_0 = 0$. Let μ be larger than the spectral radius of $\tilde{\beta}_{00}^{-\mathbf{n}}$ and let λ be larger than the spectral radius of $\tilde{\beta}_{00}^{\mathbf{n}}$ such that $\mu\lambda^2 < 1$. This is possible by the assumption of narrow spectrum in Theorem 1.1. By the above estimate on the remainder terms, for large enough i we have

$$\|\tilde{\beta}_{00}^{-i\mathbf{n}} r_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x})\| \le \mu^i \|r_i(\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x})\| \le c\mu^i \|\tilde{\alpha}_{00}^{i\mathbf{n}}\mathbf{x}\|^2 \le c\mu^i \lambda^{2i} \|\mathbf{x}\|^2.$$

By increasing i the remainder term is arbitrarily small. By the uniqueness of linear expansions, this implies that $r_0(\mathbf{x}) = 0$. ∎

Now we can write

$$f(\mathbf{x}, \mathbf{y}) = a(\mathbf{y}) + b(\mathbf{y})\mathbf{x}. \tag{6.2}$$

Calculate that

$$\tilde{\phi} \circ \tilde{\alpha}_p^{\mathbf{n}} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} a(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y}) + b(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y})(\tilde{\alpha}_{00}^{\mathbf{n}}\mathbf{x} + \tilde{\alpha}_{01}^{\mathbf{n}}\mathbf{y}) \\ \tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y} \end{pmatrix},$$
$$\tilde{\beta}_p^{\mathbf{n}} \circ \tilde{\phi} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \tilde{\beta}_{00}^{\mathbf{n}}(a(\mathbf{y}) + b(\mathbf{y})\mathbf{x}) + \tilde{\beta}_{01}^{\mathbf{n}}\mathbf{y} \\ \tilde{\beta}_{11}^{\mathbf{n}}\mathbf{y} \end{pmatrix},$$

and since these are equal, we get a functional equation

$$a(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y}) + b(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y})(\tilde{\alpha}_{00}^{\mathbf{n}}\mathbf{x} + \tilde{\alpha}_{01}^{\mathbf{n}}\mathbf{y}) = \tilde{\beta}_{00}^{\mathbf{n}}(a(\mathbf{y}) + b(\mathbf{y})\mathbf{x}) + \tilde{\beta}_{01}^{\mathbf{n}}\mathbf{y}.$$

Equivalently,

$$\begin{aligned} a(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y}) =& \tilde{\beta}_{00}^{\mathbf{n}}(a(\mathbf{y}) + \tilde{\beta}_{00}^{-\mathbf{n}}\tilde{\beta}_{01}^{\mathbf{n}}\mathbf{y} - \tilde{\beta}_{00}^{-\mathbf{n}} b(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y})\tilde{\alpha}_{01}^{\mathbf{n}}\mathbf{y}) \\ &+ (\tilde{\beta}_{00}^{\mathbf{n}} b(\mathbf{y})\mathbf{x} - b(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y})\tilde{\alpha}_{00}^{\mathbf{n}}\mathbf{x}). \end{aligned} \tag{6.3}$$

Consider what happens when $\mathbf{y}$ is held constant while $\mathbf{x}$ varies. There is no dependence on $\mathbf{x}$ on the left hand side of (6.3), whereas on the right hand side the dependence on $\mathbf{x}$ is linear. Thus in fact the last term of equation (6.3) vanishes and we obtain

$$b(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y}) = \tilde{\beta}_{00}^{\mathbf{n}} b(\mathbf{y})\tilde{\alpha}_{00}^{-\mathbf{n}}. \tag{6.4}$$

Further substituting (6.4) into (6.3) we obtain

$$a(\tilde{\alpha}_{11}^{\mathbf{n}}\mathbf{y}) = \tilde{\beta}_{00}^{\mathbf{n}}(a(\mathbf{y}) + \tilde{\beta}_{00}^{-\mathbf{n}}\tilde{\beta}_{01}^{\mathbf{n}}\mathbf{y} - b(\mathbf{y})\tilde{\alpha}_{00}^{-\mathbf{n}}\tilde{\alpha}_{01}^{\mathbf{n}}\mathbf{y}). \tag{6.5}$$

In each of the equations (6.4) and (6.5), Lemma 5.3 applies to show the terms $\tilde{\alpha}_{00}$, $\tilde{\alpha}_{01}$, $\tilde{\alpha}_{11}$, $\tilde{\beta}_{00}$, and $\tilde{\beta}_{01}$ are polynomial sums of exponentials. Thus the right hand side of both (6.4) and (6.5) consists of a vector whose entries are PSE's which depend on $\mathbf{x}$ or $\mathbf{y}$, respectively. In both cases Proposition 5.6 applies, giving

$$a(\exp(tM)\mathbf{y}) = \bar{p}_{\mathbf{y}}(t) \tag{6.6}$$

and

$$b(\exp(tM)\mathbf{y}) = \bar{q}_{\mathbf{y}}(t) \tag{6.7}$$

for some matrix M, vector-valued PSE-function $\bar{p}_{\mathbf{y}}$ and matrix-valued PSE-function $\bar{q}_{\mathbf{x}}$. Again we can choose the matrix M not to depend on the periodic point p, and to be the same matrix in these two equations.

Going back to equations (6.1) and (6.2) and using (6.6) and (6.7), we get

$$\begin{aligned}\tilde{\phi}_p\begin{pmatrix}\exp(t\,\mathrm{Id})\mathbf{x}\\ \exp(tM)\mathbf{y}\end{pmatrix} &= \begin{pmatrix}a(\exp(tM)\mathbf{y}) + e^t b(\exp(tM)\mathbf{y})\mathbf{x}\\ \mathbf{y}\end{pmatrix}\\ &= \begin{pmatrix}\bar{p}_{\mathbf{y}}(t) + e^t\bar{q}_{\mathbf{y}}(t)\mathbf{x}\\ \bar{p}_{\mathbf{y}}(t)\end{pmatrix}.\end{aligned}$$

Note that the right hand side is a vector-valued PSE-function, while the argument of $\tilde{\phi}_p$ is the matrix $\exp(tN)$ with N being the block diagonal matrix $\mathrm{diag}(\mathrm{Id}, M)$.

Note that for the proof of the above we made use of a particular basis of $\tilde{L}_{j+1}(p)$ and that a priori we do not know the regularity of the choice of this basis depending on p. However, the mere existence of a matrix N and a vector-valued PSE-function $\bar{r}_v(\cdot)$ with

$$\tilde{\phi}_p(\exp(tN)v) = \bar{r}_v(t)$$

is independent of the particular basis of $\tilde{L}_{j+1}(p)$. Thus Proposition 5.9 applies, yielding smoothness of $\tilde{\phi}$ on the fibers of $\tilde{K}_{j+1}$. Again Lemma 6.1 shows this restriction $\tilde{\phi}$ is in fact linear. This concludes the induction; it follows that $\tilde{\phi}$ is a linear e-bundle isomorphism between $\tilde{V}^-_{\mathcal{H}}$ and $\tilde{W}^-_{\mathcal{H}}$.

6.3. Smoothness of ϕ. The result of the preceding induction is that for a given half-space $\mathcal{H}$, the map $\tilde{\phi}$ is, in suitable coordinates on the fibers of the e-bundles, a linear map from $\tilde{V}^-_{\mathcal{H}}$ to $\tilde{W}^-_{\mathcal{H}}$. The map $\tilde{\phi}$ from a fiber of $\tilde{V}^-_{\mathcal{H}}$ to a fiber of $\tilde{W}^-_{\mathcal{H}}$ is locally identified by its definition with ϕ from a leaf of $V^-_{\mathcal{H}}$ to a leaf of $W^-_{\mathcal{H}}$. So we obtain that ϕ is C^∞ smooth along the leaves of $V^-_{\mathcal{H}}$ and, furthermore, the first derivative of ϕ restricted to a leaf of $V^-_{\mathcal{H}}$ is a full rank linear map.

The coarse Lyapunov foliations $V_{\mathcal{H}}^-$ are a family of transverse, linear foliations of the torus. Let $x_1, x_2, \ldots, x_d$ be a family of coordinates subordinate to these foliations. For each positive integer N, the elliptic operator

$$\frac{\partial^{2N}}{\partial x_1^{2N}} + \cdots + \frac{\partial^{2N}}{\partial x_d^{2N}}$$

applied to ϕ results in a continuous function, as all derivatives are taken along directions where ϕ is smooth. Using Theorem 5.10 we obtain ϕ is in the Sobolev space H^{2N} for each N. The intersection of these Sobolev spaces is the space of C^∞ functions. Thus ϕ is C^∞ smooth. Since it moreover has derivatives of full rank, the inverse of ϕ is also smooth. This finishes the proof of Theorem 1.1.

7. Remarks on possible extensions

There are several directions in which it may be possible to push these results.

Throughout this paper we have worked with C^∞ actions. All of our methods, however, have counterparts in the C^r category. The coarse Lyapunov foliation exists as long as stable and unstable manifolds do. The normal forms theory in finite differentiability is given in [Guy02]. The orbit closure arguments and inductive proof of smoothness of ϕ on a coarse Lyapunov leaf proceed in the normal form coordinates, and so are applicable with a result which is as smooth as the coordinate change map. The elliptic operator argument for deducing regularity for ϕ gives C^∞ smoothness in the adapted coordinates, just as in the proof above, so there is no further loss of regularity.

The narrow spectrum assumption is more troublesome, but might still be eliminated by the same approach as in this paper but using a stronger set of technical tools. The main use of this assumption is to ensure that the normal forms are linear, and thus furthermore the entire set of structures we work with are linear. Without this assumption, the normal form coordinates may only bring the maps of the action into a polynomial form. One can still produce the filtration as we do, but instead of linear structures one must deal with algebraic varieties. So far the increased technical difficulties have prevented us from carrying out this approach. If these difficulties could be surmounted, the method would almost certainly work for actions on infra-nilmanifolds as well, where the resonances which force a polynomial normal form are an unavoidable result of the nilmanifold structure. As no other methods currently exist to demonstrate differential rigidity with non-semisimple actions on infra-nilmanifolds, this is a promising direction for further development.

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